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Simple graphical constructions for the direction of shear

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Abstract—The determination of the direction and sense of the resolved shear stress on a generally-oriented plane is greatly simplified by the use of a graphical technique based on the geometry of the representation quadric of a 'reduced stress tensor'. This tensor is obtained by subtracting a constant equal to σ_3 from each of the principal stress values. The calculation requires knowledge of the orientations of the principal stress axes and the ratio of the principal stress differences. A construction for finding the direction and sense of finite shear strain on a given plane is derived in an analogous fashion. © 1998 Elsevier Science Ltd. All rights reserved

INTRODUCTION

Current attempts to use lineated fault planes as data for palaeostress reconstructions have aroused renewed interest in the theoretical relationship, derived by Wallace (1951) and Bott (1959), between the stress tensor and the direction of maximum resolved shear stress on planes of given orientation. In particular several graphical techniques have been devised for finding the shear stress direction (Lisle, 1989; Means, 1989; DePaor, 1990; Ragan, 1990; Fry, 1992; Fleischmann, 1992; Ritz, 1994).

This note describes a new method of graphically deriving the direction of shear stress which is probably the simplest yet devised. In addition it is shown that the method can be directly adapted to determine the direction of finite shear strain, a problem addressed by Schwerdtner (1998, this issue).

REPRESENTATION QUADRIC SURFACES AND DIRECTOR SURFACES

When vectors of two different types are related by linear equations, the coefficients of those equations form a second rank tensor (Nye, 1985; p. 6). A tensor, **a**, is used to convert a given or 'input' vector, **q**, into a resulting or 'output' vector, **p**, according to

$$\mathbf{p} = \mathbf{aq} \tag{1}$$

The stress tensor σ, for example, takes an input vector normal to a plane, **n**, and converts it to the stress vector **s** acting on that plane, by $\mathbf{s} = \sigma\mathbf{n}$. The differing orientations of the input and output vectors for a tensor **a** are portrayed by the representation quadric of the tensor (Nye, 1985; p. 16). This surface is described by the equation

$$a_1x^2 + a_2y^2 + a_3z^2 = \pm 1 \tag{2}$$

and is an ellipsoid or some other quadric surface depending on the signs of the principal components of the tensor, a_1, a_2 and a_3. The lengths of the semi-axes of this surface are $1/\sqrt{a_1}$, $1/\sqrt{a_2}$, $1/\sqrt{a_3}$. If a radius is drawn for the representation quadric in the direction of the input vector **q**, this will meet the surface at a point where the normal to the surface is parallel to the output vector **p** (Fig. 1a). This is known as the radius-normal property (Nye, 1985; p. 28) which converts the direction of **q** into the direction of **p**.

To convert in the converse manner, i.e. to find **q** from a known direction of **p**, the director surface (Timoshenko, 1934; p. 185) is used instead of the representation quadric. The director surface (Fig. 1b), described by the equation

$$a_1^{-1}x^2 + a_2^{-1}y^2 + a_3^{-1}z^2 = \pm 1 \tag{3}$$

has principal axes which are the reciprocal of those of the representation quadric, i.e. $\sqrt{a_1}$, $\sqrt{a_2}$, $\sqrt{a_3}$.

The radius-normal property of the director surface uses the known orientation of p (parallel to the radius) to determine the direction of q (parallel to the normal). For instance, the radius-normal property of the stress director surface can be employed to find the orientation of the normal to the plane associated with a given stress vector (Durelli *et al.*, 1958; p.147).

GRAPHICAL CONSTRUCTION FOR THE SHEAR STRESS DIRECTION

This construction is based on the representation quadric of the stress tensor (σ). Bott (1959) has shown that the direction of resolved shear stress is determined by the ratio of the principal stress differences, ϕ,

$$\phi = (\sigma_2 - \sigma_3)/(\sigma_1 - \sigma_3). \tag{4}$$

For the benefit of the present problem we can subtract a constant equal to σ_3 from each principal stress value

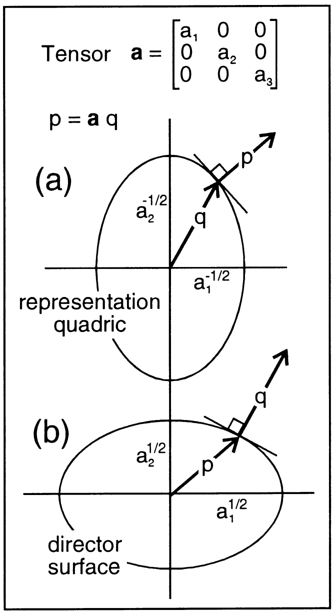

Fig. 1. The transformation of vectors **p**, **q** by means of a tensor **a**, with principal axes a_1, a_2 and a_3. (a) The radius-normal relationship of the representation quadric of **a** describes the orientations of **q** and **p**, respectively. The tangent plane is *not* the plane on which **p** acts. For simplicity, only the a_1a_2 section of the quadric is drawn. (b) The radius-normal relationship of the director surface describes the orientations of **p** and **q**, respectively. The tangent plane is the plane on which **p** acts.

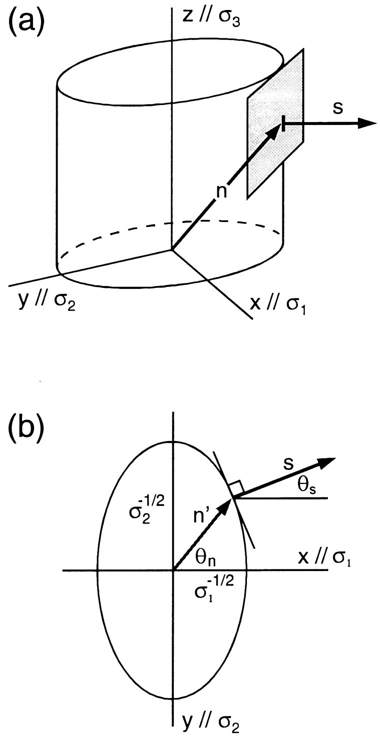

Fig. 2. (a) The representation quadric of the reduced stress tensor. The stress vector for all planes always lies parallel to the $\sigma_1\sigma_2$ plane. The shaded plane is tangential to the quadric surface. It is not the plane on which **s** acts. (b) Angular relations in the $\sigma_1\sigma_2$ plane. n' is the projection of **n**, the plane's normal. **s** is the stress vector.

without changing ϕ, i.e. the shear stress directions. The resulting 'reduced' tensor has a representation quadric with principal semi-axes of lengths

$$1/\sqrt{(\sigma_1 - \sigma_3)},\ 1/\sqrt{(\sigma_2 - \sigma_3)},\ 1/\sqrt{(\sigma_3 - \sigma_3)} = \infty, \quad (5)$$

and is therefore an elliptical or circular cylinder aligned parallel with the σ_3 axis (Fig. 2). A consequence of this special geometry of the representation quadric is that, regardless of the orientation of the plane's normal, **n**, the orientation of associated stress vector **s** always lies in the $\sigma_1\sigma_2$ plane (Fig. 2a). The exact orientation of **s** is determined by the radius-normal relationship that exists in the $\sigma_1\sigma_2$ plane between the projection of **n** (labelled n') and **s** (Fig. 2b). It can be readily shown that the angles between n' and **s** with the short axis of the elliptical section (angles θ_n and θ_s, respectively) are related by

$$\tan\theta_s / \tan\theta_n = (b/a)^2 \quad (6)$$

where a, b are the long and short semi-axes of the elliptical section, equal to $1/\sqrt{(\sigma_2 - \sigma_3)}$, $1/\sqrt{(\sigma_1 - \sigma_3)}$, respectively.

It therefore follows that

$$\tan\theta_s = \phi\tan\theta_n \quad (7)$$

Once the direction of **s** is found with use of equation (7), the direction of shear stress is finally found as the projection of **s** onto the given plane.

Summarising, the graphical construction for the direction of resolved shear stress is performed in the following steps (Fig. 3): (1) Plot the orientations of the principal stress axes σ_1, σ_2, σ_3 in stereographic projection, together with the considered plane as a great circle and pole, **n**. (2) Locate the line n' at the intersection of the σ_1 σ_2 plane with the plane containing σ_3 and **n**. (3) Measure the angle θ_n between n' and σ_1 on the stereogram and calculate θ_s using equation (7). (4) Plot **s** in the $\sigma_1\sigma_2$ plane at the angle θ_s from σ_1. (5) The resolved shear stress direction (τ) in the given plane is given by the intersection of the latter with the plane through **s** and **n**.

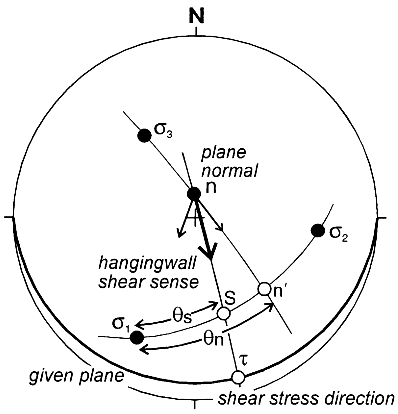

Fig. 3. The stereographic construction for the direction of shear stress, τ on a plane dipping 10° towards 180°. The arrows represent the shear sense of the hanging wall. See text for explanation.

In the example shown in Fig. 3, the direction of shear stress is determined on a plane dipping 10° towards 180°. The known principal stresses, σ_1, σ_2, σ_3 have magnitudes 40, 30 and 20 kPa and plunges 27-206, 32-096 and 45-327, respectively. The stress ratio can be found from

$$\phi = (\sigma_2 - \sigma_3)/(\sigma_1 - \sigma_3) = 0.5. \tag{8}$$

From known orientations, the angle θ_n is measured stereographically to be 58°. From equation (7), θ_s is calculated as 39°, thus allowing S to be plotted. The derived direction of shear stress pitches 76°E.

The sense of shear is determined by drawing two arrows from the plane normal **N** on the stereogram; (i) in the acute arc towards σ_1 and (ii) in the obtuse arc towards σ_3. These arrows indicate the range of feasible slip senses of the hanging wall. In the example in Fig. 3, the sense is normal with hanging wall shear to the south.

DIRECTION OF SHEAR STRAIN

Although shear strain is often used as an index of the intensity of deformation associated with simple shear, we should remind ourselves that shear strain is a generic measure of strain that records the angular changes associated with any strain history, coaxial or non-coaxial. Shear strain, like longitudinal strain is a property of individual material lines. Shear strain γ is defined as the tangent of the angle of distortion of an original right angle defined by a material line in question and the plane perpendicular to it. Figure 4(a) depicts a rock with embedded passively-deforming strain gauges, the latter with the geometry of drawing pins. After deformation the flat heads of most of the pins will not be perpendicular to their spikes (Fig. 4b). The angle of distortion is an expression of shear strain which varies according to the orientation of individual strain gauges.

Equations for shear strain magnitude γ in terms of the line's orientation relative to the principal axes of the strain ellipsoid and the principal strain magnitudes are presented in Jaeger (1962, p. 37) and Ramsay (1967, p. 128). The *direction of shear strain* of a line is defined as the direction of its projection on the plane that was originally perpendicular. In terms of our imaginary drawing pins, the direction of shear associated with each line parallel to a spike is parallel to the orthogonal projection of the spike onto the flat head (Fig. 4b). The structural relevance of the concept of direction of shear strain is discussed by Schwerdtner (1998, this issue).

In order to derive expressions for the direction of shear we consider the geometry of the finite strain ellipsoid in relation to the pre-strain unit sphere (Fig. 5). The considered line in the undeformed state is represented by a radius of the unit sphere, R (Fig. 5a). A plane perpendicular to R forms the tangent plane to the sphere drawn through the point where R meets the sphere. The normal to the tangent plane, N, is therefore parallel to R. The orientation of R (and therefore also of N) is specified by direction cosines l, m, n referred to the axes of the strain ellipsoid S_1, S_2 and S_3, respectively.

The material line corresponding to R deforms to give a vector R' with direction cosines l', m', n' where

$$l' = (lS_1)/S, \; m' = (mS_2)/S, \; n' = (nS_3)/S, \tag{9}$$

S is the stretch (the ratio of new length to old length) of the considered line and S_1, S_2 and S_3 are the principal stretches (Fig. 5b).

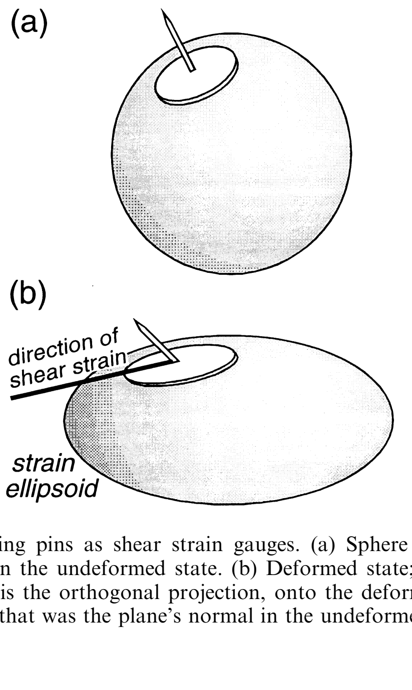

Fig. 4. Drawing pins as shear strain gauges. (a) Sphere and shear strain gauge in the undeformed state. (b) Deformed state; the direction of shear is the orthogonal projection, onto the deformed plane, of the line that was the plane's normal in the undeformed state.

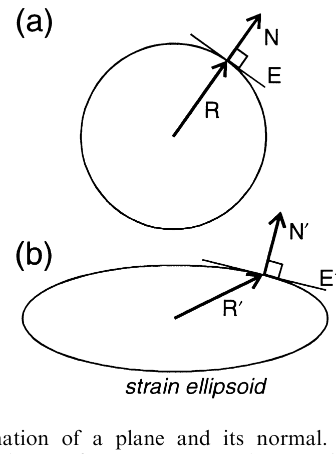

Fig. 5. Deformation of a plane and its normal. (a) Undeformed state; material plane E forms a tangent plane to the sphere. N, the normal to plane E, is parallel to radius R of the sphere. (b) Deformed state; E' is the same material plane as E and forms the tangent plane of the strain ellipsoid. The normal of the deformed plane, N', has a different orientation to the deformed material line (R') which formed the normal in the undeformed state.

The normal to a material line rotates in a similar manner but controlled by the reciprocal strain ellipsoid instead of the strain ellipsoid (March, 1932; Borradaile, 1974) so N becomes N' with direction cosines l'', m'', n'' where

$$l'' = (lS)/S_1,\ m'' = (mS)/S_2 \text{ and } n'' = (nS)/S_3 \tag{10}$$

Considering the plane containing R' and N', its normal has direction cosines, L, M, N given by Gasson (1983, p. 142) as

$$L = mn\left(\frac{S_2}{S_3} - \frac{S_3}{S_2}\right),\ M = nl\left(\frac{S_3}{S_1} - \frac{S_1}{S_3}\right),\ N = lm\left(\frac{S_1}{S_2} - \frac{S_2}{S_1}\right) \tag{11}$$

or, in terms of the orientation of the deformed tangent plane,

$$L = \frac{m''n''}{S^2}(S_2^2 - S_3^2),\ M = \frac{l''n''}{S^2}(S_3^2 - S_1^2),\ N = \frac{l''m''}{S^2}(S_1^2 - S_2^2) \tag{12}$$

We are interested in the direction of shear which is the line of intersection of this plane with the plane whose normal has direction cosines l'', m'', n''. This line with direction ratios λ:μ:v is given by the equation in Gasson (1983, p. 153):

$$\lambda : \mu : v = (Mn'' - Nm'') : (Nl'' - Ln'') : (Lm'' - Ml'') \tag{13}$$

To simplify we introduce a parameter δ, the ratio of the differences of the quadratic elongations, i.e.

$$\delta = \frac{S_2^2 - S_3^2}{S_1^2 - S_3^2} \tag{14}$$

which leads to:

$$\lambda : \mu : v = l''(l''^2 + \delta m''^2 - 1) : m''(l''^2 + \delta m''^2 - \delta) : n''(l''^2 + \delta m''^2) \tag{15}$$

From the last equation it is clear that two factors control the direction of finite shear strain: (a) The orientation of the plane on which the shear direction is to be determined, i.e. the direction cosines l'', m'', n'' and (b) the strain ellipsoid's shape parameter δ.

GRAPHICAL CONSTRUCTION FOR THE FINITE SHEAR DIRECTION

The problem in hand involves starting with a plane of specified orientation defined by its normal N' in the deformed state, and deriving the direction on that plane corresponding to the projection of line R', the deformed material line which formed the plane's normal in the undeformed state. The radius-normal construction using the finite strain ellipsoid (Fig. 5b) serves to find the direction of N'' for a known direction of R'. The strain ellipsoid is therefore the representation quadric of some tensor **c** which converts vector R' into vector N' i.e. $\mathbf{N}' = \mathbf{cR}'$.

The strain ellipsoid has principal semi-axes S_1, S_2, S_3 and therefore the tensor **c** it represents must have principal values $1/S_1^2$, $1/S_2^2$, $1/S_3^2$. This tensor is the Cauchy deformation tensor (Truesdell and Toupin, 1960; p. 257; Means, 1976; p. 198). Our present task however is to determine R' from known N', rather than the converse. Therefore we need to use the director surface of **c** rather than its representation quadric. It follows that this director surface will have principal semi-axes of length $1/S_1$, $1/S_2$ and $1/S_3$ oriented parallel to the principal axes of the strain ellipsoid, S_1, S_2 and S_3, respectively. We now employ the same device as used to solve the shear stress problem. Equations (14) and (15) show that if the deformation tensor is modified by subtracting a constant equal to S_3^2 from each principal quadratic elongation, we will not alter the directions of shear strain. The modified deformation tensor has principal values of $1/(S_1^2–S_3^2)$, $1/(S_2^2–S_3^2)$, $1/(S_3^2–S_3^2) = \infty$, and therefore the associated director surface is an elliptical cylinder with principal semi-axes of lengths of $1/\sqrt{(S_1^2–S_3^2)}$, $1/\sqrt{(S_2^2–\text{i}\ S_3^2)}$, ∞.

The resulting graphical construction (Fig. 6) is directly analogous to that for the direction of shear stress. That procedure should be followed with appropriate substitution of symbols [σ_1, σ_2, σ_3, ϕ, n, s, θ_n, θ_s] with [S_1, S_2, S_3, δ, N', R', θ_n, θ_R]. Figure 6 illustrates the construction of the direction of shear strain on a plane dipping 10° towards 180° for principal stretches $S_1 = 1.25$, $S_2 = 0.90$, $S_3 = 0.25$ with orientations 27-206, 32-096 and 45-327, respectively. The direction of shear pitches 76°E.

Incidentally, the directions labelled γ_{xn} and γ_{zn} represent the shear directions for $\delta = 0$ and $\delta = 1$, re-

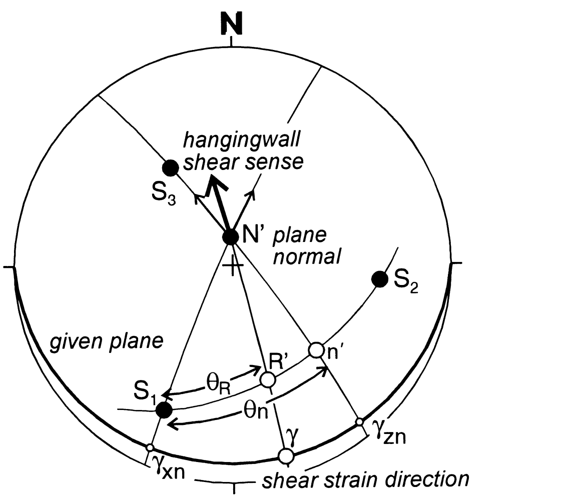

Fig. 6. Construction for the direction of shear strain, γ, on a plane dipping 10° towards 180°. The arrows represent the shear sense of the hanging wall. See text for explanation. γ_{xn} and γ_{zn} are the directions referred to by Schwerdtner (1998); they define the bounds on admissible orientations of γ.

spectively. As such they represent the extreme orientations of the shear direction; they define bounds for admissible shear directions (Schwerdtner, 1998).

The sense of shear is determined by drawing two arrows from the plane normal N' on the stereogram; (i) in the acute arc towards S_3 and (ii) in the obtuse arc towards S_1. These arrows indicate the senses for the range of feasible shear directions of the hanging wall. In the example in Fig. 6, the sense is reversed with hanging wall shear to the north.

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REFERENCES

Borradaile, G. M. (1974) Bulk finite tectonic strain estimates from the deformation of neptunian dykes. *Tectonophysics* **22,** 127–139.

Bott, M. H. P. (1959) The mechanisms of oblique slip faulting. *Geological Magazine* **96,** 109–117.

DePaor, D. G. (1990) The theory of shear stress and shear strain on planes inclined to the principal directions. *Journal of Structural Geology* **12,** 923–927.

Durelli, A. J., Phillips, E. A. and Tsao, C. H. (1958) *Introduction to the Theoretical and Experimental Analysis of Stress and Strain*. McGraw-Hill, New York.

Fleischmann, K. H. (1992) A graphical construction for shear stress on a fault surface. *Journal of Structural Geology* **14,** 499–502.

Fry, N. (1992) Direction of shear. *Journal of Structural Geology* **14,** 253–255.

Gasson, P. C. (1983) *Geometry of Spatial Forms*. Ellis Horwood, Chichester.

Jaeger, J. C. (1962) *Elasticity, Fracture and Flow*. Methuen, London.

Lisle, R. J. (1989) A simple construction for shear stress. *Journal of Structural Geology* **11,** 493–495.

March, A. (1932) Mathematische Theorie der Regulung nach der Korngestalt bei Affiner Deformation. *Zeitschrift für Kristallographie*. **81,** 285–297.

Means, W. D. (1976) *Stress and Strain*. Springer Verlag, New York.

Means, W. D. (1989) A construction for shear stress on a generally-oriented plane. *Journal of Structural Geology* **11,** 625–627.

Nye, J. F. (1985) *Physical Properties of Crystals*. Oxford University Press, London.

Ragan, D. M. (1990) Direction of shear. *Journal of Structural Geology* **12,** 929–931.

Ramsay, J. G. (1967) *Folding and Fracturing of Rocks*. McGraw-Hill, New York.

Ritz, J. F. (1994) Determining the slip vector by graphical construction: use of a simplified representation of the stress tensor. *Journal of Structural Geology* **16,** 737–741.

Schwerdtner, W. M. (1998) Graphic derivation of the local sense of shear strain components in stretched walls of lithotectonic boundaries. *Journal of Structural Geology* **20,** 957–967.

Timoshenko, S. (1934) *Theory of Elasticity*. McGraw-Hill, New York.

Truesdell, C. and Toupin, R. A. (1960) The classical field theories. In *Encyclopedia of Physics*, ed. S. Flügge, pp. 226–793. Springer Verlag, Berlin, **3**.

Wallace, R. E. (1951) Geometry of shearing stress and relation to faulting. *Journal of Geology* **59,** 118–130.